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The Pfaff lattice on symplectic matrices

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Abstract
The Pfaff lattice is an integrable system arising from the SR-group factorization in an analogous way to how the Toda lattice arises from the QR-group factorization. In our earlier paper (Kodama and Pierce 2007 *Int. Math. Res. Not.* (arXiv:0705.0510)), we studied the Pfaff lattice hierarchy for the case where the Lax matrix is defined to be a lower Hessenberg matrix. In this paper we deal with the case of a symplectic lower Hessenberg Lax matrix, this forces the Lax matrix to take a 2×2 block tridiagonal shape. We then show that the odd members of the Pfaff lattice hierarchy are trivial, while the even members are equivalent to the indefinite Toda lattice hierarchy defined in Kodama and Ye (1996 *Physica* D **91** 321–39). This is analogous to the case of the Toda lattice hierarchy in relation to the Kac–van Moerbeke system. In the case with the initial matrix having only real or imaginary eigenvalues, the fixed points of the even flows are given by 2×2 block diagonal matrices with zero diagonals. We also consider a family of skew-orthogonal polynomials with a symplectic recursion relation related to the Pfaff lattice and find that they are succinctly expressed in terms of orthogonal polynomials appearing in the indefinite Toda lattice.

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1. Introduction

The Toda lattice hierarchy of nonlinear differential equations may be formulated as the continuous limit of the QR-algorithm for diagonalizing a symmetric matrix [8, 19, 20]. There exists a number of generalizations of the QR-algorithm for diagonalizing matrices while preserving different symmetries; the HR and SR-algorithms are two that we will consider in this paper [4, 7, 10, 19]. In particular, the SR-algorithm relies on SR-factorizations whose goal is to factor an invertible matrix into a symplectic matrix and an upper triangular matrix

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[6]. The Pfaff lattice hierarchy was recently introduced to describe the partition functions of the orthogonal and symplectic ensembles of random matrices and, equivalently, to give the evolution of skew-orthogonal polynomials (see [1, 3]). The Pfaff lattice hierarchy may be viewed as a continuous limit of the SR-algorithm for diagonalizing a lower Hessenberg matrix, and in fact was originally discovered in this context [22]. We showed in our recent work [16] that the Pfaff lattice is an integrable system in the Arnold–Liouville sense, and in the case that the initial conditions are not lower Hessenberg matrices from the symplectic algebra, $\mathfrak{sp}(n)$, we showed that the fixed points of the Pfaff flows with an initial condition having real distinct eigenvalues are given by 2×2 block diagonal matrices with zero diagonals.

In this paper we will consider the Pfaff lattice flow with an initial condition given by a matrix from the symplectic algebra $\mathfrak{sp}(n)$. The Lie algebra $\mathfrak{sp}(n)$ is the set of $2n \times 2n$ matrices satisfying $JL + L^T J = 0$ with

$$J = \mathrm{diag}_2\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = I_n \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $\mathrm{diag}_2(A)$ denotes the 2×2 block diagonal matrix of A, i.e. $\mathrm{diag}_2(A) = \mathrm{diag}_2(A_1, \ldots, A_2)$ with the 2×2 blocks A_j in the diagonal of A, and I_n is the $n \times n$ identity matrix.

Remark 1.1. Matrices in the symplectic algebra, $\mathfrak{sp}(n)$, are called Hamiltonian in the linear algebra and control theory literature; we will avoid this nomenclature to avoid confusion with the Hamiltonians which are constants of motion and appear throughout the paper. The term symplectic algebra is also the one commonly used in the literature on the Pfaff lattice and in representation theory [11].

The spectrum of a real symplectic matrix comes in three types:

(a) pairs of real eigenvalues $(z, -z)$;
(b) pairs of imaginary eigenvalues $(z, -z)$ and
(c) quadruples of complex eigenvalues $(z, \bar{z}, -z, -\bar{z})$.

Symplectic matrices appear in a number of control theory problems most notably in solving the real algebraic Riccati equation (see e.g. [9]). Our J is a permutation of the traditional J, say $\tilde{J}$, used in the literature on the Hamiltonian eigenvalue problem, i.e. $\tilde{J} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \otimes I_n$. We use this J because of its connection to the Pfaff lattice which is defined in lower Hessenberg matrices (see below).

The Pfaff lattice can be viewed as an $\mathfrak{sp}$-version of the Toda lattice; it has the following form with $\mathfrak{sp}$-projection [1, 16]:

$$\frac{\partial L}{\partial t} = [\pi_{\mathfrak{sp}}(\nabla H), L].$$

The Pfaff lattice is associated with the Lie–Poisson structure induced by the splitting $\mathfrak{sl}(2n) = \mathfrak{k} \oplus \mathfrak{sp}(n)$. Recall that the splitting gives the following projections of an element $X \in \mathfrak{sl}(2n, \mathbb{R})$:

$$\pi_{\mathfrak{k}}(X) = X_- - J(X_+)^T J + \tfrac{1}{2}\big(X_0 - JX_0^T J\big) \tag{1.1}$$

$$\pi_{\mathfrak{sp}}(X) = X_+ + J(X_+)^T J + \tfrac{1}{2}\big(X_0 + JX_0^T J\big), \tag{1.2}$$

where X_0 is the 2×2 block diagonal part of X and X_+ (resp. X_-) is the 2×2 block strictly upper (resp. lower) triangular part of X. In particular, we have

$$\mathfrak{k} = \left\{ \begin{pmatrix} h_1 I_2 & 0_2 & \cdots & 0_2 \\ \star & h_2 I_2 & \cdots & 0_2 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & h_n I_2 \end{pmatrix} : \sum_{k=1}^{n} h_k = 0 \right\}, \tag{1.3}$$

where I_2 is the 2×2 identity matrix, and 0_2 is the 2×2 zero matrix.

As in the case of the Toda lattice hierarchy, one can also define the Pfaff lattice hierarchy (see [16] and also section 3)

$$\frac{\partial L}{\partial t_j} = [\pi_{\mathfrak{sp}}(\nabla H_j), L], \qquad \text{for} \quad j = 1, \ldots, 2n-1, \tag{1.4}$$

where $H_j = \frac{1}{j+1} \operatorname{tr}(L^{j+1})$ and L is a Hessenberg matrix given by, for example with $n = 3$,

$$L = \begin{pmatrix} 0 & c_1 & 0 & 0 & 0 & 0 \\ * & b_1 & a_1 & 0 & 0 & 0 \\ * & * & -b_1 & c_2 & 0 & 0 \\ * & * & * & b_2 & a_2 & 0 \\ * & * & * & * & -b_2 & c_3 \\ * & * & * & * & * & 0 \end{pmatrix}. \tag{1.5}$$

The c_k are invariants of the Pfaff lattice, and in the traditional Pfaff lattice considered by Adler *et al* [1], the c_k are chosen to be 1. It was shown in [16] that the c_k are Casimirs of the Pfaff lattice. In [16] it was also shown that for L with distinct eigenvalues, the Pfaff lattice hierarchy is an integrable system in the Arnold–Liouville sense. In addition it was shown that if L is not symplectic and has distinct real eigenvalues, then as $t_j \to \infty$ for any j, there is a diagonal matrix $P(t_j)$ such that the normalized matrix $\hat{L}(t_j) := P(t_j)L(t_j)P(t_j)^{-1}$ approaches a 2×2 block upper triangular matrix, where the diagonal blocks are sorted by the size of z^j for the real eigenvalues z appearing in each. In this paper, we deal with the Pfaff lattice when the initial matrix is symplectic and discuss the dynamical structure in connection with the Toda lattice.

A natural question at this point is what happens when we restrict the Pfaff lattice variables to matrices which are both symplectic and lower Hessenberg. The first consequence is that such matrices have the form

$$L_0 = \left(\begin{array}{cc|cc|c|c} 0 & c_1 & 0 & 0 & \cdots & 0_2 \\ d_1 & 0 & a_1 & 0 & & \\ \hline 0 & 0 & 0 & c_2 & \cdots & 0_2 \\ a_1 & 0 & d_2 & 0 & & \\ \hline \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \vdots \\ \hline \multicolumn{2}{c|}{0_2} & \multicolumn{2}{c|}{0_2} & \cdots & \begin{matrix} 0 & c_n \\ d_n & 0 \end{matrix} \end{array}\right), \tag{1.6}$$

that is, L_0 has a 2×2 block tridiagonal form, $L_0 = (l_{i,j})_{1 \leqslant i,j \leqslant n}$ with 2×2 block matrices $l_{i,j}$ having $l_{i,j} = 0_2$ for $|i - j| > 1$ and

$$l_{k,k} = \begin{pmatrix} 0 & c_k \\ d_k & 0 \end{pmatrix}, \qquad l_{k,k+1} = l_{k+1,k} = \begin{pmatrix} 0 & 0 \\ a_k & 0 \end{pmatrix},$$

where $c_k = \pm 1$. We call a matrix in this form 'H^S-tridiagonal'. This is analogous to restricting the Toda lattice variables to a matrix which is tridiagonal and skew-symmetric (i.e. in $\mathfrak{so}(n)$); a situation we will remark on in more detail in section 4.

We will show that the even Pfaffian flow of an H^S-tridiagonal matrix converges to a sequence of block diagonal matrices with the following shapes:

(a) a 2×2 block for each real pair of eigenvalues $(z, -z)$,

(b) a 2×2 block for each imaginary pair of eigenvalues $(z, -z)$,

(c) a 4×4 block for each quadruple of complex eigenvalues $(z, \bar{z}, -z, -\bar{z})$.

In addition, for the t_{2j} flow, the blocks are sorted by the size of $\mathrm{Re}(z^{2j})$. In the appendix, we will give a simple proof that the SR-algorithm is given by the integer evaluation of the flow generated by the Pfaff lattice introduced in [1] (see also [16]).

The present paper is organized as follows. In section 2 we review the SR-factorization which is closely associated with the Pfaff lattice hierarchy of equations. We then show, in section 3, that the odd members of the hierarchy of the Pfaff lattice (1.4) on symplectic matrices are trivial, while the even members of the hierarchy on H^S-tridiagonal matrices are equivalent to the indefinite Toda hierarchy introduced in [17] (theorem 3.1). This is similar to the case of the Toda lattice in the connection with the Kac–van Moerbeke system. In contrast with the symmetric Toda lattice hierarchy, the indefinite Toda hierarchy experiences blow ups, where some entries of the matrix approach infinity. Our result then implies that for generic H^S-tridiagonal matrices, the Pfaff lattice has blow ups. In section 4, we give a further discussion on the equivalence between the Pfaff lattice on H^S-tridiagonal matrix and the indefinite Toda lattice defined in [17]. We begin by reviewing the indefinite Toda lattice, and then show that the τ-functions for the Pfaff lattice are the same as those for the indefinite Toda lattice (theorem 4.1). We then examine the families of skew-orthogonal polynomials appearing in the Pfaff lattice for an H^S-tridiagonal matrix. We show that these polynomials are related to the orthogonal polynomials in the indefinite Toda lattice (theorem 4.2). Finally, we remark on the asymptotic behavior of the even Pfaff lattice flows of an H^S-tridiagonal matrix. In particular we show that initial conditions with complex eigenvalues, or with complex eigenvectors, will result in a blow up.

2. Background on SR-factorization

The Pfaff lattice equation is intimately connected to the SR-factorization of an invertible matrix and so we collect here some relevant facts on this factorization. With the Lie subalgebra $\mathfrak{k}$ given by (1.3) in the Lie algebra splitting $\mathfrak{sl}(2n) = \mathfrak{k} \oplus \mathfrak{sp}(n)$, we define $G_{\mathfrak{k}}$ to be the Lie group with Lie algebra $\mathfrak{k}$

$$G_{\mathfrak{k}} := \left\{ \begin{pmatrix} \alpha_1 I_2 & 0_2 & \cdots & 0_2 \\ \star & \alpha_2 I_2 & \cdots & 0_2 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & \alpha_n I_2 \end{pmatrix} : \prod_{j=1}^{n} \alpha_j = 1 \right\}.$$

We also define the group $\tilde{G} \supset G_{\mathfrak{k}}$ to be the group of real invertible lower 2×2 block triangular matrices with *free* invertible diagonal blocks, that is, $\tilde{G}$ is a parabolic subgroup of lower 2×2 block matrices of $SL(2n, \mathbb{R})$.

We define the Pfaffian of a skew-symmetric matrix m by the recursive formula

$$\mathrm{pf}(m) = \sum_{j=1}^{2n} (-1)^{i+j+1} m_{ij} \mathrm{pf}(m_{\hat{i}\hat{j}}),$$

where $m_{\hat{i}\hat{j}}$ is found by deleting the ith and jth rows and columns of m, where i is any row, and the initial value for the recursion is given by

$$\mathrm{pf}\begin{pmatrix} 0 & m_{12} \\ -m_{12} & 0 \end{pmatrix} = m_{12}.$$

Then, for example, with the skew symmetric matrix

$$m = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} \\ & 0 & m_{23} & m_{24} \\ & & 0 & m_{34} \\ & & & 0 \end{pmatrix},$$

we have

$$\mathrm{pf}(m) = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}.$$

Throughout this paper, we leave blank the lower triangular part of skew-symmetric matrices.

We will use the SR-factorization of $g \in SL(2n, \mathbb{R})$ introduced by Bunse–Gerstner (theorem 3.8 in [6], see also [5]).

Theorem 2.1. *Let $g \in SL(2n, \mathbb{R})$ and $M = gJg^T$. Then the factorization*

$$g = rs$$

with $r \in \tilde{G}$ and $s \in Sp(n, \mathbb{R})$ exists if and only if the Pfaffian of the $2k \times 2k$ upper left submatrix of M, denoted by M_{2k}, does not vanish, i.e.

$$\mathrm{pf}(M_{2k}) \neq 0, \qquad 1 \leqslant k \leqslant n-1.$$

As a consequence there is a dense set of matrices in $SL(2n, \mathbb{R})$ for which this decomposition exists. If we restrict the factorization to using elements of $G_{\mathfrak{k}} \subset \tilde{G}$ the theorem becomes

Theorem 2.2. *Let $g \in SL(2n, \mathbb{R})$ and $M = gJg^T$. Then the factorization*

$$g = rs$$

with $r \in G_{\mathfrak{k}}$ and $s \in Sp(n, \mathbb{R})$ exists if and only if all the Pfaffians $\mathrm{pf}(M_{2k})$ satisfy the positivity condition,

$$\mathrm{pf}(M_{2k}) > 0, \qquad 1 \leqslant k \leqslant n-1.$$

This implies that there is an open set of matrices in $SL(2n, \mathbb{R})$ for which this decomposition exists.

Theorem 3.7 in [6] shows that the set of matrices in $SL(2n, \mathbb{C})$ with SR-factorization is neither open nor dense in $SL(2n, \mathbb{C})$. For this reason we will restrict our consideration to the real groups for now.

In remark 3.9 in [6] it is noted that for the choice of subgroups of $SL(2n, \mathbb{R})$ in that paper, the decomposition $g = rs$ is not unique, as one may pass a factor from

$$\tilde{G} \cap Sp(n, \mathbb{R}) = \{\mathrm{diag}_2(A_1, A_2, \ldots, A_n) : \det(A_j) = 1\, \forall j\},$$

between r and s. In the case of SR factorization with R restricted to $G_{\mathfrak{k}}$ we find that factorizations are unique up to a factor from

$$G_{\mathfrak{k}} \cap Sp(n, \mathbb{R}) = \{\mathrm{diag}_2(\pm I_2, \pm I_2, \ldots, \pm I_2)\}.$$

One notes that $G_{\mathfrak{k}}$ is in fact made up of 2^n connected components, and if we further restrict R to the component of $G_{\mathfrak{k}}$ containing the identity matrix we obtain unique factorizations. Equivalently we obtain unique factorizations by asking that the diagonal elements of r be positive. In contrast the group $\tilde{G}$ is connected.

The Pfaffians defined in theorem 2.1 are also called the τ-functions, which play the fundamental role in the Pfaff lattice hierarchy [1, 16], i.e. for $M = gJg^T$ and M_{2k}, the $2k \times 2k$ upper left submatrix of M,

$$\tau_{2k} := \mathrm{pf}(M_{2k}), \qquad k = 0, 1, \ldots, n, \tag{2.1}$$

with $\tau_0 = 1$. Derivatives of the τ-functions for $g(\mathbf{t}) := \exp\left(\sum_{j=1}^{2n-1} t_j L_0^j\right)$, with $L(0) = L_0$, generate the matrix entries of $L(\mathbf{t})$ in the form (1.5), and the solution of the Pfaff lattice hierarchy. For example, we have [1, 16] (see section 3)

$$a_k(\mathbf{t}) = a_k(0)\frac{\sqrt{\tau_{2k+2}(\mathbf{t})\tau_{2k-2}(\mathbf{t})}}{\tau_{2k}(\mathbf{t})}, \qquad b_k(\mathbf{t}) = \frac{\partial}{\partial t_1}\ln \tau_{2k}(\mathbf{t}). \tag{2.2}$$

We also have the following lemma:

Lemma 2.1. *Let $g = rs$ be a $2n \times 2n$ matrix where $r \in G_{\mathfrak{k}}$ and $s \in Sp(n, \mathbb{R})$. Then the 2×2 block diagonal entries of r, $\mathrm{diag}_2(r) = \mathrm{diag}_2(r_1 I_2, \ldots, r_n I_2)$, are expressed by the τ-functions*

$$r_k = \sqrt{\frac{\tau_{2k}}{\tau_{2k-2}}}.$$

Proof. Since $M = gJg^T = rsJs^Tr^T = rJr^T$, we have $\tau_{2k} = \mathrm{pf}(M_{2k}) = \prod_{j=1}^{k} r_j^2$, which leads to the formula. □

A necessary condition for the existence of the factorization is that $\tau_{2k} = \mathrm{pf}(M_{2k})$ must be positive (theorem 2.2); lemma 2.1 and the convention that $\tau_0 = 1$ explain this condition. There are explicit algorithms for carrying out SR-factorization and they have many features in common with those developed for QR-factorization; we refer the reader to [9, 6].

3. Pfaff lattice hierarchy

In [1], Adler, Horozov and van Moerbeke introduced the Pfaff lattice hierarchy to describe the partition functions of the Gaussian orthogonal and symplectic ensembles of random matrices (GOE and GSE) and to describe the evolution of the recursion relations of skew-orthogonal polynomials. The finite Pfaff lattice hierarchy is a set of Hamiltonian flows on $\mathfrak{sl}(2n, \mathbb{R})$ with the Lie–Poisson structure induced by the splitting $\mathfrak{sl}(2n, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{sp}(n)$, and it is defined as follows (see also [16]). For any $X, Y \in \mathfrak{sl}(2n)$ we first define the $\mathcal{R}$-bracket

$$[X, Y]_{\mathcal{R}} = [\mathcal{R}X, Y] + [X, \mathcal{R}Y],$$

where $\mathcal{R}$ is the R-matrix given by $\mathcal{R} = \frac{1}{2}(\pi_{\mathfrak{k}} - \pi_{\mathfrak{sp}})$. Then the Lie–Poisson bracket for any functions F and H on $\mathfrak{sl}^*(2n) \cong \mathfrak{sl}(2n)$ is defined by

$$\{F, H\}_{\mathcal{R}}(L) = \langle L, [\nabla F, \nabla H]_{\mathcal{R}}\rangle,$$

where $\langle A, B\rangle = \mathrm{tr}(AB)$ and ∇F is defined by $\langle X, \nabla F\rangle = \frac{\mathrm{d}}{\mathrm{d}\epsilon} F(L+\epsilon X)|_{\epsilon=0}$. The Pfaff lattice with respect to a Hamiltonian function $H(L)$ is defined by

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \{H(L), L\}_{\mathcal{R}}(L).$$

In particular, if the Hamiltonian is Sp-invariant, one can write

$$\{H(L), L\}_{\mathcal{R}}(L) = [\pi_{\mathfrak{sp}}(\nabla H(L)), L].$$

The traditional Hamiltonians giving the Pfaff lattice are $\frac{1}{k+1}\mathrm{tr}(L^{k+1})$ [16]. The Pfaff lattice hierarchy is then defined by

$$\frac{\partial L}{\partial t_k} = [\pi_{\mathfrak{sp}}(L^k), L] \qquad k = 1, \ldots, 2n-1. \tag{3.1}$$

Thus the t_k-flow of the Pfaff lattice hierarchy is associated with the Hamiltonian $H_k = \frac{1}{k+1}\mathrm{tr}(L^{k+1})$. Each flow is solved by the following factorization procedure: Factor

$$\exp(t\nabla H(L(0))) = R(t)S(t) \tag{3.2}$$

with the initial matrix $L(0)$ using SR-factorization with R in the connected component of $G_{\mathfrak{k}}$ containing the identity and $S \in Sp(n)$, and then the solution is given by

$$L(t) = R(t)^{-1}L(0)R(t) = S(t)L(0)S(t)^{-1}. \tag{3.3}$$

With this in mind, one sees that as in the case of the QR-algorithm with the Toda lattice, the SR-algorithm is given by integer evaluations of the Pfaff flow with Hamiltonian $H(L) = \mathrm{tr}(L\ln(L) - L)$ (see the appendix for some details).

In [16], we showed that the Pfaff lattice hierarchy (3.1) is an integrable system in the Arnold–Liouville sense. Normalizing the matrix L of the form (1.5) by

$$\hat{L} = PLP^{-1}$$

with the diagonal matrix P in the 2×2 block form

$$P = \mathrm{diag}_2\left(I_2, a_1 I_2, (a_1a_2)I_2, \ldots, \left(\prod_{j=1}^{n-1} a_j\right) I_2\right),$$

where $\hat{L}$ is a lower Hessenberg matrix with 1's on the super diagonal. Then we showed that if L is not symplectic and all the eigenvalues are *real* and *distinct*, then $t_j \to \infty$, $\hat{L}(t_j)$ converges to a 2×2 block upper triangular matrix such that the diagonal blocks are sorted by the size of z^j.

Solutions of the Pfaff lattice hierarchy are generated by the τ-functions (introduced as obstructions to the SR-factorization in the previous section). They are found by the following procedure. We first consider the factorization of $g(\mathbf{t}) := \exp\left(\sum_{j=1}^{2n-1} t_j L_0^j\right)$ with $L_0 = L(0)$ (see (3.2)):

$$g(\mathbf{t}) = R(\mathbf{t})S(\mathbf{t}) \qquad \text{with} \qquad R(\mathbf{t}) \in G_{\mathfrak{k}},\ S(\mathbf{t}) \in Sp(n).$$

Then the skew-symmetric matrix $M(\mathbf{t}) = g(\mathbf{t})Jg^T(\mathbf{t})$ becomes

$$M(t) = R(\mathbf{t})JR^T(\mathbf{t}).$$

Since $R \in G_{\mathfrak{k}}$, we have

$$\mathrm{diag}_2(R) := \mathrm{diag}(r_1 I_2, r_2 I_2, \ldots, r_n I_2).$$

Then the τ-functions τ_{2k} defined in (2.1) can be written by

$$\tau_{2k}(\mathbf{t}) = \mathrm{pf}(M(\mathbf{t})_{2k}) = \prod_{j=1}^{k} r_j(\mathbf{t})^2.$$

Then from (3.3), i.e. $R(\mathbf{t})L(\mathbf{t}) = L(0)R(\mathbf{t})$, we have

$$a_k(\mathbf{t}) = a_k(0)\frac{r_{k+1}(\mathbf{t})}{r_k(\mathbf{t})} = a_k(0)\frac{\sqrt{\tau_{2k+2}(\mathbf{t})\tau_{2k-2}(\mathbf{t})}}{\tau_{2k}(\mathbf{t})},$$

which gives the a_k's in (2.2) (see also [1, 16]).

3.1. Odd Pfaff flows of symplectic matrices

Let us first show that if $L(0)$ is a symplectic matrix then it is a fixed point of the odd members of the Pfaff lattice hierarchy:

Lemma 3.1. *For $L \in \mathfrak{sp}(n)$, the odd power L^{2j-1} is also symplectic.*

Proof. Being symplectic is equivalent to $JLJ = L^T$. Suppose that $JL^{2j-3}J = (L^T)^{2j-3}$. Then we have

$$JL^{2j-1}J = JL^{2j-3}JJLJJLJ = (L^T)^{2j-3}L^TL^T = (L^T)^{2j-1}$$

so that the lemma is true by induction. □

Therefore, $\pi_{\mathfrak{sp}}(L^{2j-1}) = L^{2j-1}$ for $L \in \mathfrak{sp}(n)$; hence, the odd members of the Pfaff hierarchy become trivial, i.e.

$$\frac{\partial L}{\partial t_{2j-1}} = [L^{2j-1}, L] = 0.$$

Note in particular that all b_k in (2.2) vanish, which is consistent with the form L in theorem A.1, that is, the diagonal elements are all zero.

We note that this also happens for the Toda lattice hierarchy. Recall that the Toda lattice equation for a symmetric matrix is based on the Lie algebra splitting $\mathfrak{sl}(n) = \mathfrak{b} \oplus \mathfrak{so}(n)$, where $\mathfrak{b}$ is the set of upper triangular matrices. With the pairing $\langle A, B\rangle := \mathrm{tr}(AB)$ for $A, B \in \mathfrak{sl}(n)$, we have $\mathfrak{sl}(n) \cong \mathfrak{sl}(n)^* = \mathfrak{b}^* \oplus \mathfrak{so}(n)^*$ where $\mathfrak{b}^* \cong \mathfrak{so}(n)^\perp = \mathrm{Sym}(n)$ and $\mathfrak{so}(n)^* \cong \mathfrak{b}^\perp = \mathfrak{n}$. Here $\mathrm{Sym}(n)$ is the set of symmetric matrices, and $\mathfrak{n}$ is the set of strictly upper triangular matrices. Then the Lie–Poisson bracket for the functions F, G on $\mathrm{Sym}(n)$ is defined by

$$\{F, H\}(L) = \langle L, [\nabla F, \nabla H]\rangle \qquad \text{for} \quad L \in \mathrm{Sym}(n).$$

The Toda lattice hierarchy for $L \in \mathrm{Sym}(n)$ is then defined by

$$\frac{\partial L}{\partial t_j} = \{H_j, L\}(L) = [\pi_{\mathfrak{so}}(\nabla H_j), L]. \tag{3.4}$$

With the Hamiltonian functions $H_j(L) = \frac{1}{j+1}\mathrm{tr}(L^{j+1})$ for $j = 1, \ldots, n-1$, the differential equation for L becomes

$$\frac{\partial L}{\partial t_j} = [\pi_{\mathfrak{so}}(L^j), L], \tag{3.5}$$

with $\pi_{\mathfrak{so}}(X)$ meaning the projection of the matrix X on the $\mathfrak{so}(n)$ component.

Now if we extend the Toda lattice equation for the general $L \in \mathfrak{sl}(n, \mathbb{R})$, an analogue of lemma 3.1 is true with $\mathfrak{so}(n)$ instead of $\mathfrak{sp}(n)$. Then, for $L \in \mathfrak{so}(n)$ (i.e. skew-symmetric), the odd members of the hierarchy become trivial. However, the even powers of L are symmetric, and the even members of the generalized Toda hierarchy for L are equivalent to symmetric

tridiagonal Toda lattices. This case includes the Kac–van Moerbeke system [14]. Consider for example the following $2n \times 2n$ skew-symmetric tridiagonal matrix:

$$L = \begin{pmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ -\alpha_1 & 0 & \alpha_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\alpha_{2n-2} & 0 & \alpha_{2n-1} \\ 0 & \cdots & 0 & -\alpha_{2n-1} & 0 \end{pmatrix}.$$

Then the Kac–van Moerbeke hierarchy may be expressed by $\frac{\partial L}{\partial t_{2j}} = [\pi_{\mathfrak{so}}(L^{2j}), L]$, where the first member gives

$$\frac{\partial \alpha_k}{\partial t_2} = \alpha_k\left(\alpha_{k-1}^2 - \alpha_{k+1}^2\right), \qquad k = 1, \ldots, 2n-1,$$

with $\alpha_0 = \alpha_{2n} = 0$. We then note that the square L^2 is a symmetric matrix,

$$L^2 = T^{(1)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + T^{(2)} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.6}$$

where $T^{(i)}$, for $i = 1, 2$, are $n \times n$ symmetric tridiagonal matrices given by

$$T^{(i)} = \begin{pmatrix} b_1^{(i)} & a_1^{(i)} & 0 & \cdots & 0 \\ a_1^{(i)} & b_2^{(i)} & a_2^{(i)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n-1}^{(i)} & a_{n-1}^{(i)} \\ 0 & 0 & \cdots & a_{n-1}^{(i)} & b_n^{(i)} \end{pmatrix},$$

with $a_k^{(1)} = \alpha_{2k-1}\alpha_{2k}$, $b_k^{(1)} = -\alpha_{2k-2}^2 - \alpha_{2k-1}^2$, $a_k^{(2)} = \alpha_{2k}\alpha_{2k+1}$ and $b_k^{(2)} = -\alpha_{2k-1}^2 - \alpha_{2k}^2$. Then one can show that each $T^{(i)}$ gives the symmetric Toda lattice, that is, the equation $\frac{\partial L}{\partial t_{2j}} = [\pi_{\mathfrak{so}}(L^{2j}), L]$ splits into two Toda lattices:

$$\frac{\partial T^{(i)}}{\partial t_{2j}} = [\pi_{\mathfrak{so}}(T^{(i)})^j, T^{(i)}], \qquad \text{for} \quad i = 1, 2.$$

The equations for $T^{(i)}$ are connected by the Miura-type transformation, with the functions $\left(a_k^{(i)}, b_k^{(i)}\right)$, through the Kac–van Moerbeke variables α_k (see [13]).

We now show that the even Pfaff flows on an H^S-tridaigonal matrix have a similar structure.

3.2. Even Pfaff flows of symplectic matrices

We first show that the H^S-tridiagonal form (1.6) is invariant under the even members of the Pfaff hierarchy. Then we show that the Pfaff flow for the matrix L in the form (1.6) is related to the indefinite Toda lattice defined in [17]. The indefinite Toda lattice is a continuous version of the HR algorithm [10] and is defined as follows (see also section 4 for some details): let C be the $n \times n$ diagonal matrix given by

$$C = \mathrm{diag}(c_1, c_2, \ldots, c_n),$$

where c_k's are the elements appearing in the diagonal blocks of (1.6), and they are invariant under the Pfaff flow (see [16]). We also define $\tilde{T}$ by $\tilde{T} := CT$ with the triadiagonal matrix (A.1), i.e.

$$\tilde{T} = \begin{pmatrix} c_1 d_1 & c_1 a_1 & \cdots & \cdots & 0 \\ c_2 a_1 & c_2 d_2 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{n-1} d_{n-1} & c_{n-1} a_{n-1} \\ 0 & 0 & \cdots & c_n a_{n-1} & c_n d_n \end{pmatrix}. \tag{3.7}$$

Then the indefinite Toda lattice hierarchy is defined by

$$\frac{\partial \tilde{T}}{\partial t_{2j}} = [B_j, \tilde{T}], \tag{3.8}$$

where $B_j := [\tilde{T}^j]_+ - [\tilde{T}^j]_-$. Here the $[\tilde{T}^j]_\pm$ represent the projections on the strictly upper (+) and strictly lower (−) triangular parts of the matrix $\tilde{T}^j$ (sometimes we write $\tilde{T}^j_\pm = [\tilde{T}^j]_\pm$).

With L in the H^S-tridiagonal form (1.6), we first note

$$L^2 = \tilde{T} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \tilde{T}^T \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.9}$$

from which we have, for any even power,

$$L^{2j} = \tilde{T}^j \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (\tilde{T}^j)^T \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 3.1. We remark here that equation (3.9) is similar to (3.6); this structure is fundamental and implies that our problem will have a similar structure to that of the Kac–van Moerbeke lattice.

Then using the projection (1.2), we obtain

$$\pi_{\mathfrak{sp}}(L^{2j}) = B_j \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - B_j^T \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.10}$$

with $B_j = \tilde{T}^j_+ - \tilde{T}^j_-$. With (3.10), one can easily show that the H^S-tridiagonal form is invariant under the even members of the Pfaff lattice hierarchy. It is also immediate to see from (3.10) that the $2m$th member of the Pfaff lattice hierarchy (1.4) in terms of L^2 is equivalent to the indefinite Toda lattice (3.8). Thus we have the following theorem:

Theorem 3.1. *The even Pfaff flow for $L \in \mathfrak{sp}(n, \mathbb{R})$ in the H^S-tridiagonal matrix form (1.6),*

$$\frac{\partial L}{\partial t_{2m}} = [\pi_{\mathfrak{sp}}(L^{2m}), L],$$

is equivalent to the indefinite Toda flow for $\tilde{T} = CT$ with the symmetric tridiagonal matrix T of (A.1) and $C := \mathrm{diag}(c_1, c_2, \ldots, c_n)$,

$$\frac{\partial \tilde{T}}{\partial t_{2m}} = [B_m, \tilde{T}], \tag{3.11}$$

where $B_m = \tilde{T}^m_+ - \tilde{T}^m_-$.

If $c_k = 1$ for all k, then $\tilde{T} = T$ is symmetric and $B_m = \pi_{\mathfrak{so}}(T^m)$. Theorem 3.1 then says that the $2m$th Pfaff flow is just the traditional symmetric tridiagonal mth flow of the Toda lattice hierarchy. In terms of the SR-algorithm in proposition A.4, we obtain as a corollary

of theorem 3.1 that the even-integer iterates L_{2k} of the SR-algorithm are equivalent to the $\tilde{T}_k$ iterates of the HR-algorithm. This result is shown directly in [4]; the key is the structure of equation (3.9). With the permuted J matrix used in [4], i.e. $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \otimes I_n$, the right-hand side of relation (3.9) takes the form

$$\begin{pmatrix} \tilde{T} & 0_n \\ 0_n & \tilde{T}^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \tilde{T} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tilde{T}^T.$$

4. Pfaff lattice versus indefinite Toda lattice

Here we give a further discussion on the equivalence between the Pfaff lattice hierarchy on the H^S-tridiagonal matrix and the indefinite Toda lattice hierarchy introduced in [17].

4.1. The indefinite Toda lattice

Let us begin with a brief description of the indefinite Toda lattice defined by (3.8) for a matrix $\tilde{T} = CT$ with a symmetric matrix T of the form (A.1), i.e.

$$\frac{\partial \tilde{T}}{\partial t} = [B, \tilde{T}], \qquad \text{with} \qquad B = \tilde{T}_+ - \tilde{T}_-.$$

The solution $\tilde{T}(t)$ with the initial matrix $\tilde{T}_0$ can be solved by the HR-factorization,

$$\tilde{g} = \exp(t\tilde{T}_0) = r(t)h(t),$$

where r is a lower triangular matrix and h satisfies $hCh^T = C$ (recall that if $C = I_n, h \in SO(n)$). It is then easy to show that the solution $\tilde{T}(t)$ is obtained by

$$\tilde{T}(t) = r(t)^{-1}\,\tilde{T}_0\,r(t) = h(t)\,\tilde{T}_0\,h(t)^{-1}. \tag{4.1}$$

We note that the entries a_k in $\tilde{T}$ are expressed in terms of the diagonal elements of r, say $\mathrm{diag}(r) = \mathrm{diag}(r_1, \ldots, r_n)$,

$$a_k(t) = a_k(0)\frac{r_{k+1}(t)}{r_k(t)}.$$

To find those diagonal entries from $\tilde{g}$, one considers the following matrix called the moment matrix:

$$M^{\mathrm{Toda}} := \tilde{g}C\tilde{g}^T = rCr^T. \tag{4.2}$$

Then the matrix r can be found by the Cholesky factorization method. Now the τ-functions are defined by

$$\tau_k^{\mathrm{Toda}} = \det\left(M_k^{\mathrm{Toda}}\right),$$

where M_k^{Toda} is the $k \times k$ upper left submatrix of M^{Toda}. With (4.2), we have $\tau_k^{\mathrm{Toda}} = \prod_{j=1}^k c_j\, r_j^2$, which gives $c_k r_k^2 = \tau_k/\tau_{k-1}$. Then the entries a_k of $\tilde{T}$ can be expressed in terms of the τ-functions:

$$a_k(t) = a_k(0)\sqrt{\frac{c_k}{c_{k+1}}}\,\frac{\left[\tau_{k+1}^{\mathrm{Toda}}(t)\,\tau_{k-1}^{\mathrm{Toda}}(t)\right]^{1/2}}{\tau_k^{\mathrm{Toda}}(t)}. \tag{4.3}$$

As will be shown in the next section (see theorem 4.1 below), those are the same as the formulas for the a_k's in (2.2) given in terms of the τ-functions of the Pfaff lattice.

Let us also discuss the orthogonal functions appearing in the indefinite Toda lattice. First we note that the Lax form (3.8) is given by the compatibility of the equations

$$\tilde{T}\Phi = \Phi D \qquad \text{and} \qquad \frac{\partial \Phi}{\partial t_{2j}} = B_j \Phi,$$

where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with the eigenvalues λ_k and $\Phi = (\phi_i(\lambda_j))_{1\leqslant i,j\leqslant n}$, the eigenmatrix. As the orthogonality condition of the eigenvectors with a normalization, we have

Lemma 4.1.

$$\Phi C \Phi^T = C.$$

Proof. We consider the following relation which is equivalent to the orthogonality relation,

$$\Phi^T C^{-1} \Phi = C^{-1}.$$

(This is sometimes called the second orthogonality relation.) To show this, we note

$$\begin{aligned}\lambda_j \phi(\lambda_i)^T C^{-1} \phi(\lambda_j) &= \phi(\lambda_i)^T C^{-1} \tilde{T} \phi(\lambda_j) = \phi(\lambda_i)^T T \phi(\lambda_j) \\ &= (\tilde{T}\phi(\lambda_i))^T C^{-1} \phi(\lambda_j) = \lambda_i \phi(\lambda_i)^T C^{-1} \phi(\lambda_j).\end{aligned}$$

Since we assume $\lambda_i \neq \lambda_j$, the matrix $\Phi^T C^{-1} \Phi$ is diagonal. We also note that this matrix is invariant under the flow, i.e. $\partial(\Phi^T C^{-1}\Phi)/\partial t = 0$. Normalizing the diagonal matrix gives the result. □

We then define the inner product for functions $f(\lambda)$ and $g(\lambda)$,

$$\langle fg \rangle_{\mathrm{Toda}} := \sum_{k=1}^{n} f(\lambda_k) g(\lambda_k) c_k, \tag{4.4}$$

which defines a discrete measure $\mathrm{d}\mu(\lambda) = \sum_{k=1}^{n} c_k \delta(\lambda - \lambda_k)\, \mathrm{d}z$. Lemma 4.1 implies

$$\langle \phi_i \phi_j \rangle_{\mathrm{Toda}} = c_i \delta_{ij}.$$

With the orthogonality relation, the moment matrix M^{Toda} can be expressed by

$$M^{\mathrm{Toda}} = \Phi_0\, \mathrm{e}^{2tD} C \Phi_0^T = \left(\left\langle \phi_i^0 \phi_j^0\, \mathrm{e}^{2tz} \right\rangle_{\mathrm{Toda}}\right)_{1\leqslant i,j\leqslant n}, \tag{4.5}$$

where Φ_0 is the initial eigenmatrix with the eigenvector $\phi_i^0 = \phi_i^0(\lambda) = \phi_i(\lambda, 0)$.

In terms of the initial eigenmatrix Φ_0, we can find an explicit form of $\Phi(\mathbf{t})$ for the indefinite Toda lattice hierarchy with $\mathbf{t} = (t_2, t_4, \ldots, t_{2n})$ (see theorem 1 in [17] and theorem 2 in [15]):

Lemma 4.2. *The eigenvector $\phi(\lambda, \mathbf{t}) = (\phi_1(\lambda, \mathbf{t}), \ldots, \phi_n(\lambda, \mathbf{t}))^T$ can be expressed as*

$$\phi_k(\lambda, \mathbf{t}) = \frac{c_k^{1/2}\, \mathrm{e}^{\xi(\lambda,\mathbf{t})}}{\left[\tau_k^{\mathrm{Toda}}(\mathbf{t})\, \tau_{k-1}^{\mathrm{Toda}}(\mathbf{t})\right]^{1/2}} \begin{vmatrix} m_{1,1}(\mathbf{t}) & \cdots & m_{1,k-1}(\mathbf{t}) & \phi_1^0(\lambda) \\ m_{2,1}(\mathbf{t}) & \cdots & m_{2,k-1}(\mathbf{t}) & \phi_2^0(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ m_{k,1}(\mathbf{t}) & \cdots & m_{k,k-1}(\mathbf{t}) & \phi_{k-1}^0(\lambda) \end{vmatrix}, \tag{4.6}$$

where $m_{i,j}(\mathbf{t}) := \left\langle \phi_i^0 \phi_j^0\, \mathrm{e}^{2\xi(\lambda,\mathbf{t})} \right\rangle_{\mathrm{Toda}}$ with $\xi(\lambda, \mathbf{t}) = \sum_{k=1}^{n} \lambda^k t_{2k}$ and the τ-functions are given by

$$\tau_k^{\mathrm{Toda}} = \det\left(M_k^{\mathrm{Toda}}\right) = |(m_{i,j})_{1\leqslant i,j\leqslant k}|. \tag{4.7}$$

Proof. We give here the proof of lemma 4.2 for the sake of completeness, though it is a relatively simple modification of the results of [17, 15]. With the factorization $e^{\xi(\tilde{T}_0,\mathbf{t})} = r(\mathbf{t})h(\mathbf{t})$, we have $\tilde{T}(\mathbf{t}) = h(\mathbf{t})\tilde{T}_0 h(\mathbf{t})^{-1}$ (see (4.1)). Then from lemma 4.1, we obtain

$$\Phi(\mathbf{t}) = h(\mathbf{t})\Phi_0.$$

Now from the factorization with $\tilde{T}\Phi = \Phi D$, we have

$$\Phi(\mathbf{t}) = r(\mathbf{t})^{-1} e^{\xi(\tilde{T}_0,\mathbf{t})}\Phi_0 = r(\mathbf{t})^{-1}\Phi_0\, e^{\xi(D,\mathbf{t})},$$

which implies

$$\phi_i(\lambda,\mathbf{t}) \in \mathrm{Span}\left\{\phi_1^0(\lambda)\, e^{\xi(\lambda,\mathbf{t})}, \ldots, \phi_i^0(\lambda)\, e^{\xi(\lambda,\mathbf{t})}\right\}, \qquad i = 1,\ldots,n. \tag{4.8}$$

Then using the Gram–Schmidt orthogonalization method with the inner product (4.4), we obtain the result. □

If we set $\phi_1^0(\lambda) = 1$ and consider the semi-infinite lattice with $n = \infty$, $\phi_k(\lambda,\mathbf{t})$ can be expressed in the following elegant form in terms of only the τ-functions:

Proposition 4.1. *The orthogonal eigenfunctions $\phi_k(\lambda)$ can be expressed in terms of τ-functions,*

$$\phi_k(\lambda,\mathbf{t}) = \frac{c_k^{1/2}\, e^{\xi(\lambda,\mathbf{t})}}{\left[\tau_k^{\mathrm{Toda}}(\mathbf{t})\tau_{k-1}^{\mathrm{Toda}}(\mathbf{t})\right]^{1/2}}\tau_{k-1}^{\mathrm{Toda}}\left(\mathbf{t} - \frac{1}{2}[\lambda^{-1}]\right)\lambda^{k-1},$$

where $\xi(\lambda,\mathbf{t}) = \sum_{k=1}^{\infty}\lambda^k t_{2k}$ and $\tau_j^{\mathrm{Toda}}\left(\mathbf{t} - \frac{1}{2}[\lambda^{-1}]\right) = \tau_j^{\mathrm{Toda}}\left(t_2 - \frac{1}{2\lambda}, \ldots, t_{2n} - \frac{1}{2n\lambda^n}, \ldots\right)$.

Proof. We give the complete proof here, though one can see that it is a relatively simple modification of that in [2] for the case $c_k = 1$, so that the manuscript is self-contained. In this more general case, the proof follows from formula (4.6) with $\xi(\lambda,\mathbf{t}) = \sum_{k=1}^{\infty}\lambda^k t_{2k}$. First we note that using (4.8), one can replace $\phi_j^0(\lambda)$ in (4.6) by λ^{j-1} with $\phi_1^0(\lambda) = 1$, which gives $m_{i,j}(\mathbf{t}) = \langle\lambda^{i+j-2}\, e^{2\xi(\lambda,\mathbf{t})}\rangle_{\mathrm{Toda}}$. This implies that $m_{i,j}(\mathbf{t}) = \partial^{i+j-2}m_{1,1}(\mathbf{t})/\partial t_1^{i+j-2}$; hence, $\tau_k^{\mathrm{Toda}}(\mathbf{t})$ is given by the Hankel determinant form, $\tau_k^{\mathrm{Toda}} = |(h_{i+j-1})_{1\leqslant i,j\leqslant k}|$ with $h_{i+j-1} := m_{i,j}$. Since $h_j(\mathbf{t})$ is a linear combination of the exponential function $E_i(\mathbf{t}) = e^{2\xi(\lambda_i,\mathbf{t})}$ and $E_i\left(\mathbf{t} - \frac{1}{2}[\lambda^{-1}]\right) = \left(1 - \frac{\lambda_i}{\lambda}\right)E_i(\mathbf{t})$, we have

$$h_j\left(\mathbf{t} - \frac{1}{2}[\lambda^{-1}]\right) = h_j(\mathbf{t}) - \frac{1}{\lambda}h_{j+1}(\mathbf{t}).$$

Then it is easy to see that

$$\tau_k^{\mathrm{Toda}}\left(\mathbf{t} - \frac{1}{2}[\lambda^{-1}]\right)\lambda^k = \begin{vmatrix} h_1(\mathbf{t}) & h_2(\mathbf{t}) & \cdots & h_k(\mathbf{t}) & 1 \\ h_2(\mathbf{t}) & h_3(\mathbf{t}) & \cdots & h_{k+1}(\mathbf{t}) & \lambda \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_k(\mathbf{t}) & h_{k+1}(\mathbf{t}) & \cdots & h_{2k-1}(\mathbf{t}) & \lambda^{k-1} \\ h_{k+1}(\mathbf{t}) & h_{k+2}(\mathbf{t}) & \cdots & h_{2k}(\mathbf{t}) & \lambda^k \end{vmatrix},$$

which gives the determinant in (4.7) in terms of $\tau_{k-1}^{\mathrm{Toda}}$.

For a finite n, one can see that $\tau_n^{\mathrm{Toda}}\left(\mathbf{t} - \frac{1}{2}[\lambda^{-1}]\right)\lambda^n \big/ \tau_n^{\mathrm{Toda}}(\mathbf{t})$ is proportional to the polynomial $\prod_{k=1}^{n}(\lambda - \lambda_k)$ and we have $\phi_{n+1}(\lambda) = 0$, which is just the characteristic polynomial of L. □

With proposition 4.1, we will show a further example of the close relation between the orthogonal functions in the indefinite Toda lattice and the skew-orthogonal functions in the Pfaff lattice for the symplectic matrix of (1.6).

4.2. The τ-functions of the Pfaff and Toda lattices

We show here that the τ-functions which generate the solutions of the Pfaff lattice equations are equivalent to the τ-functions which generate the solutions of the indefinite Toda lattice hierarchy. Let us recall that the τ-functions of the Pfaff lattice are defined by (2.1), i.e.

$$\tau_{2k} = \mathrm{pf}(M_{2k}),$$

where M_{2k} is the $2k \times 2k$ upper left submatrix of $2n \times 2n$ skew-symmetric matrix M given by

$$M := gJg^T \qquad \text{with} \qquad g = \mathrm{e}^{\xi(L_0^2, \mathbf{t})},$$

where $\xi\left(L_0^2, \mathbf{t}\right) = \sum_{k=1}^{n} t_{2k} L_0^{2k}$ with the initial matrix $L(0) = L_0$. Then our goal is to show the following theorem:

Theorem 4.1. *The τ-functions of the even-flows of the Pfaff lattice hierarchy with the initial matrix $L(0)$ in the H^S-tridiagonal form of (1.6) are related to the τ-functions of the indefinite Toda lattice hierarchy by*

$$\tau_{2k} = \frac{1}{c_1 \cdots c_k} \tau_k^{\mathrm{Toda}}.$$

Proof. We first note from (3.9) that the entries of $g = \mathrm{e}^{\xi(L_0^2, \mathbf{t})}$ are expressed by those of $\tilde{g} = \mathrm{e}^{\xi(\tilde{T}_0, \mathbf{t})}$,

$$g = \tilde{g} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \tilde{g}^T \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\tilde{T}_0 = CT_0$ with a symmetric matrix T_0, $C^{-1}\tilde{g}$ is symmetric, i.e. $C^{-1}\tilde{g} = \tilde{g}^T C^{-1}$. Then for $H = C^{-1}M^{\mathrm{Toda}} = C^{-1}\tilde{g}C\tilde{g}^T = \tilde{g}^T\tilde{g}^T$ and $M = gJg^T$, we have

$$M = H \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + H^T \otimes \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{4.9}$$

Let H_k be the $k \times k$ upper left submatrix of H, and M_{2k} be the submatrix defined above. Then with the permutation matrix,

$$P = [e_1, e_3, \ldots, e_{2k-1}, e_2, e_4, \ldots, e_{2k}],$$

where e_k's are the standard basis (column) vectors on $\mathbb{R}^{2k}$, one can transform M_{2k} to

$$P^T M_{2k} P = \begin{pmatrix} 0_k & H_k \\ -H_k^T & 0_k \end{pmatrix} = \begin{pmatrix} H_k & 0_k \\ 0_k & I_k \end{pmatrix} P^T J P \begin{pmatrix} H_k^T & 0_k \\ 0_k & I_k \end{pmatrix}. \tag{4.10}$$

We recall the Pfaffian identity for a skew-symmetric matrix B,

$$\mathrm{pf}(A^T BA) = \det(A)\mathrm{pf}(B),$$

which with (4.10) implies that

$$\det(P)\mathrm{pf}(M_{2k}) = \det(P)\det\begin{pmatrix} H_k & 0_k \\ 0_k & I_k \end{pmatrix}\mathrm{pf}(J).$$

We finish the proof by noting that P is invertible and $\mathrm{pf}(J) = 1$ giving

$$\mathrm{pf}(M_{2k}) = \det\begin{pmatrix} H_k & 0_k \\ 0_k & I_k \end{pmatrix} = \det(H_k) = \frac{1}{c_1 \cdots c_k}\det\left(M_k^{\mathrm{Toda}}\right).$$

□

4.3. Skew-orthogonal and orthogonal functions

We show here that the eigenvectors of the matrix L in the form (1.6) define a family of skew-orthogonal functions, and then discuss an explicit relation between those skew-orthogonal functions and the orthogonal functions appearing in the indefinite Toda lattice, i.e. those in proposition 4.1.

As in the case of the Toda lattice, the Pfaff lattice hierarchy (3.1) is given by the compatibility of the linear equations

$$L\Psi = \Psi\Lambda \qquad \text{and} \qquad \frac{\partial\Psi}{\partial t_{2j}} = \pi_{\mathfrak{sp}}(L^{2j})\,\Psi, \tag{4.11}$$

where L is symplectic and in the form of (1.6), and Λ is the eigenvalue matrix,

$$\Lambda := D \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

with $D = \mathrm{diag}(z_1, \ldots, z_n)$. (Recall that the eigenvalues of L consist of the pairs $(z_k, -z_k)$ for $k = 1, \ldots, n$, and we assume here that they are all distinct.) Then we have:

Lemma 4.3. *The eigenmatrix Ψ satisfies the skew-orthogonal relation,*

$$\Psi^T J \Psi = \mathcal{K} J,$$

with a diagonal matrix with nonzero constants κ_k's in the form,

$$\mathcal{K} = \mathrm{diag}_2\,(\kappa_1 I_2, \kappa_2 I_2, \ldots, \kappa_n I_2)\,.$$

Proof. From (4.11) with $B := \pi_{\mathfrak{sp}}(L^{2j})$ and $t = t_{2j}$, we have

$$\begin{aligned} \frac{\partial}{\partial t}(\Psi^T J \Psi) &= \Psi^T B^T J \Psi + \Psi^T J B \Psi \\ &= \Psi^T (B^T J + J B)\Psi = 0, \end{aligned}$$

where we have used $B \in \mathfrak{sp}(n)$.

Now we note the following equation for the eigenvectors $\psi(z_k)$,

$$\begin{aligned} z_j \psi(z_k)^T J \psi(z_j) &= \psi(z_k)^T J L \psi(z_j) \\ &= -\psi(z_k)^T L^T J \psi(z_j) \qquad (\because JL + L^T J = 0) \\ &= -(L\psi(z_k))^T J \psi(z_j) \\ &= -z_k \psi(z_k)^T J \psi(z_j). \end{aligned}$$

This implies

$$(z_k + z_j)\psi(z_k)^T J \psi(z_j) = 0.$$

Noting the order of the eigenvalues in $\Lambda = \mathrm{diag}(-z_1, z_1, \ldots, -z_n, z_n)$, so that $\Psi^T J \Psi$, which is skew-symmetric, is a multiple of J of the form $\mathcal{K}$ with nonzero constant κ_k's. □

From lemma 4.3, we have

$$\Psi\,\mathcal{K}^{-1}\,J\,\Psi^T = J, \tag{4.12}$$

which implies the skew-orthogonality relation for the functions $\psi_k(z)$ in the eigenvector $\psi(z) = (\psi_0(z), \psi_1(z), \ldots, \psi_{2n-1}(z))$ of L:

$$\begin{cases} \langle \psi_{2j}, \psi_{2k} \rangle = \langle \psi_{2j+1}, \psi_{2k+1} \rangle = 0, \\ \langle \psi_{2j}, \psi_{2k+1} \rangle = \langle \psi_{2j-1}, \psi_{2k} \rangle = \delta_{jk}, \end{cases} \qquad \text{for} \quad j \leqslant k. \tag{4.13}$$

Here the skew-inner product $\langle f, g\rangle$ is defined by

$$\langle f, g\rangle := \sum_{k=1}^{n} [f(-z_k)g(z_k) - f(z_k)g(-z_k)]\,\kappa_k^{-1}. \tag{4.14}$$

Now we claim

Theorem 4.2. *The skew-orthogonal eigenfunctions $\psi_k(z, \mathbf{t})$ satisfying (4.13) can be given by*

$$\begin{cases} \psi_{2k}(z, \mathbf{t}) = \phi_{k+1}(\lambda, \mathbf{t}) \\ \psi_{2k+1}(z, \mathbf{t}) = c_{k+1}^{-1} z\phi_{k+1}(\lambda, \mathbf{t}) \end{cases} \qquad \text{with} \quad \lambda = z^2,$$

where $\phi_k(\lambda, \mathbf{t})$'s are the orthogonal eigenfunctions appearing in the indefinite Toda lattice (see proposition 4.1) and the measure of the inner product $\langle\cdot,\cdot\rangle$ is given by $\kappa_k = 2z_k c_k^{-1}$.

Before proving the theorem, we recall the following proposition which gives the skew-orthogonal functions appearing in the semi-infinite Pfaff lattice:

Proposition 4.2. *The eigenvector $\psi(z, \mathbf{t}) = (\psi_0(z, \mathbf{t}), \psi_1(z, \mathbf{t}), \ldots)^T$ for the matrix L of (1.5) with $n = \infty$ can be expressed in terms of τ-functions:*

$$\begin{cases} \psi_{2k}(z, \mathbf{t}) = \dfrac{e^{\xi(z,\mathbf{t})}}{[\tau_{2k}(\mathbf{t})\tau_{2k+2}(\mathbf{t})]^{1/2}}\,\tau_{2k}(\mathbf{t} - [z^{-1}])z^{2k} \\ \psi_{2k+1}(z, \mathbf{t}) = \dfrac{c_{k+1}^{-1} e^{\xi(z,\mathbf{t})}}{[\tau_{2k}(\mathbf{t})\tau_{2k+2}(\mathbf{t})]^{1/2}}\left(z + \dfrac{\partial}{\partial t_1}\right)\tau_{2k}(\mathbf{t} - [z^{-1}])z^{2k}, \end{cases}$$

where $\xi(z, \mathbf{t}) = \sum_{k=1}^{\infty} z^k t_k$ and $\tau_{2k}(\mathbf{t} - [z^{-1}]) = \tau_{2k}\big(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z}, \ldots\big)$.

This proposition appears as theorem 3.2 in [3] for the case $c_k = 1$. In this more general case the additional factor of a c_{k+1}^{-1} is present to ensure that the recursion relation L has $L_{2k,2k+1} = c_{k+1}$. This entry of the Pfaff variable is a Casimir of the Pfaff lattice equations and so is fixed. In the definition of the skew-orthogonal functions, it corresponds to a choice of the ratio between leading coefficients of the polynomial parts of ψ_{2k} and ψ_{2k+1}.

Now we prove theorem 4.2.

Proof. First we show that $\psi_k(z, \mathbf{t})$'s in the theorem are the same as those in proposition 4.2, when the matrix L is in an H^S-tridiagonal form (1.6). To show this, we recall theorem 4.1, i.e.

$$\tau_{2k}(\mathbf{t}) = \frac{1}{c_1 \cdots c_k}\tau_k^{\text{Toda}}(\mathbf{t}).$$

Since the odd flows are trivial, we have $\mathbf{t} = (t_2, t_4, \ldots, t_{2n})$. Then substituting this relation into $\psi_k(z, \mathbf{t})$ in proposition 4.2, it is straightforward to show the equations in the theorem.

Now we show the skew-orthogonality relation (4.3). We only check the case $\langle\psi_{2j}, \psi_{2k+1}\rangle = \delta_{jk}$, and the others are trivial with the form of the inner product:

$$\begin{aligned} \langle\psi_{2j}, \psi_{2k+1}\rangle &= \sum_{i=1}^{n}[\psi_{2j}(-z_i)\psi_{2k+1}(z_i) - \psi_{2j}(z_i)\psi_{2k+1}(-z_i)]\frac{c_i}{2z_i} \\ &= c_{k+1}^{-1}\sum_{i=1}^{n}\phi_{j+1}(\lambda_i)\phi_{k+1}(\lambda_i)c_i = \delta_{jk}, \end{aligned}$$

where we have used the orthogonal relation $\Phi C \Phi^T = C$ in lemma 4.1. □

We also note that the moment matrix $M(\mathbf{t})$ for the Pfaff lattice has the similar form as the $M^{\mathrm{Toda}}(\mathbf{t})$ for the indefinite Toda lattice given in (4.5):

$$\begin{aligned} M(\mathbf{t}) &= \mathrm{e}^{\xi(L_0^2,\mathbf{t})} J \mathrm{e}^{\xi(L_0^2,\mathbf{t})^T} = \Psi_0\, \mathrm{e}^{\xi(\Lambda^2,\mathbf{t})} \Psi_0^{-1} J \Psi_0^{-T} \mathrm{e}^{\xi(\Lambda^2,\mathbf{t})} \Psi_0^T \\ &= \Psi_0\, \mathrm{e}^{2\xi(\Lambda^2,\mathbf{t})} \mathcal{K}^{-1} J \Psi_0^T = \left(\left\langle \psi_i^0, \psi_j^0\, \mathrm{e}^{2\xi(z^2,\mathbf{t})}\right\rangle\right)_{0\leqslant i,j\leqslant 2n-1}. \end{aligned}$$

Comparing with (4.9), the entries $m_{ij} := \left\langle \psi_i^0, \psi_j^0 \mathrm{e}^{2\xi(z^2,\mathbf{t})}\right\rangle$ are given by $m_{ij} = -m_{ji}$ with

$$m_{2j-1,2k} = -m_{2k,2j-1} \qquad \text{for} \quad j \leqslant k,$$

and zero for all other cases, i.e. $m_{2j-1,2k-1} = m_{2j,2k} = 0$. With this structure of the moment matrix, one can give a direct proof of theorem 4.2 without proposition 4.2; however, this might be a less elegant approach.

Remark 4.1. The skew-inner product of (4.14) is closely related to the skew-inner product for GSE-Pfaff lattice, which is defined as

$$\langle f, g\rangle_{\mathrm{GSE}} := \sum_{k=1}^{n} \{f, g\}(z_{2k}) c_k,$$

where $\{f, g\}(z) = f'(z)g(z) - g'(z)f(z)$ with $f'(z) = \mathrm{d}f(z)/\mathrm{d}z$. This inner product may be obtained by the following inner product in the limit $z_{2k-1} \to z_{2k}$ (see [16]), i.e.

$$\langle f, g\rangle_{\mathrm{GSE}} = \lim \sum_{k=1}^{n} \frac{f(z_{2k-1})g(z_{2k}) - f(z_{2k})g(z_{2k-1})}{z_{2k} - z_{2k-1}} c_k.$$

In the case of (4.14), we instead take the limit $z_{2k-1} \to -z_{2k}$.

Also in the case of the continuous measure, the inner product (4.14) can be written by

$$\langle f, g\rangle = \int_{\Sigma} [f(-z)g(z) - f(z)g(-z)]\, c(z^2)\, \mathrm{d}z,$$

where $\Sigma = \mathbb{R}_+ \cup i\mathbb{R}_+$ oriented from $\mathrm{i}\infty$ to 0 and then to ∞. To reduce this expression back to the discrete case, we take

$$\begin{aligned} c(z^2)\,\mathrm{d}z\Big|_{\Sigma} &= \sum_{k=1}^{n} \delta(z^2 - z_k^2)\,\mathrm{d}z\Big|_{\Sigma} = \sum_{k=1}^{n} \delta((z - z_k)(z + z_k))\,\mathrm{d}z\Big|_{\Sigma} \\ &= \sum_{k=1}^{n} \left[\frac{1}{2z_k}\delta(z - z_k) + \frac{1}{2z_k}\delta(z + z_k)\right]\mathrm{d}z\Big|_{\Sigma} = \sum_{k=1}^{n} \frac{1}{2z_k}\delta(z - z_k)\,\mathrm{d}z\Big|_{\Sigma}, \end{aligned}$$

with $z_k \in \Sigma$, and where we used the property $\delta(az) = \frac{1}{|a|}\delta(z)$ for $a \in \mathbb{R}$ and $\delta(az) = \frac{1}{i|a|}\delta(z)$ for $a \in \mathrm{i}\mathbb{R}$. The last equality is because we have restricted to the positive and positive imaginary square roots with $z_k \in \Sigma$. The result is that this discrete inner product will agree with (4.14).

4.4. Asymptotics of the even Pfaff flows

In this final section, we mention the asymptotic behavior of the even Pfaff lattice based on the results of the indefinite Toda lattice discussed in [17, 18]. Theorem 3.1 implies that the Pfaff flow on the H^S-tridiagonal matrix L_0 with $c_k = \pm 1$ is equivalent to the indefinite Toda flow on $\tilde{T} = CT$ given by (3.7). This system was studied in detail in [17, 18]. It is a version of the Toda lattice which uses HR-factorization in place of QR-factorization, see [10]. The goal of

HR-factorization is to write an element $g \in SL(n, \mathbb{R})$ as $g = rh$, where r is lower triangular and h satisfies

$$hCh^T = C.$$

For $g(t_{2j}) := \exp(t_{2j}\tilde{T}(0)^j)$, the HR-factorization $g(t_{2j}) = r(t_{2j})h(t_{2j})$ gives the solution of the t_{2j}-flow of the Pfaff lattice

$$\tilde{T}(t_{2j}) = r^{-1}(t_{2j})\tilde{T}(0)r(t_{2j}) = h(t_{2j})\tilde{T}(0)h^{-1}(t_{2j}).$$

Thus the indefinite Toda lattice is a continuous version of the HR-algorithm [10].

The indefinite Toda flow may experience a blow up, where some of the entries reach infinity in finite time [18]. A blow up occurs when one of the τ functions becomes 0. In our case this is precisely when one of the $\tau_{2k} = 0$. The initial conditions for a blow-up are characterized by

Theorem 4.3 (Theorem 3 in [17]). *If $\tilde{T}(0)$ possesses non-real eigenvalues or non-real eigenvectors while $\tau_{2k}(0) \neq 0$, then $\tilde{T}(t)$ blows up to infinity in finite time.*

The second possibility occurs if the c_k do not all have the same sign. The case when the eigenvalues of $\tilde{T}$ are real and the c_k do not all have the same sign is studied in section 4 of [18]. It is shown that every flow (considered in both directions $t_{2j} \to \pm\infty$) contains a blow up. We may then compactify the flows by adding infinite points representing the blow ups. It is then shown that the fixed points of the compactified flows are diagonal matrices. In other words $a_k \to 0$ and $d_k \to c_k z^2_{\sigma(k)}$ as $t_{2j} \to \infty$ for some permutation $\sigma \in \mathcal{S}_n$, the symmetric group of order n. When the eigenvalues of $\tilde{T}$ are real we may count the total number of blow ups on the flow. If the number of changes of sign in the sequence c_k is m then the number of blow ups is $m(n-m)$ [18].

There is a condition under which the indefinite Toda flow, with $c_k = -1$ for some k, does not contain a blow up in the positive t direction. A lower triangular matrix is called lower triangular totally positive if all its non-trivial minors are positive. It was shown in [12] that if the lower component of the LU-factorization of the eigenvector matrix of $\tilde{T}_0$ is lower triangular totally positive then there are no blow ups on the indefinite Toda flow for $t > 0$.

If L_0 has only real and imaginary eigenvalues, then $\tilde{T}$ has real eigenvalues. By theorem 3.1 we find that the flow may be continued through a blow up, if it occurs, and that this compactified flow converges:

$$L(t_{2j}) \to \mathrm{diag}_2\left(\begin{pmatrix} 0 & c_1 \\ c_1 z^2_{\sigma(1)} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & c_n \\ c_n z^2_{\sigma(n)} & 0 \end{pmatrix}\right),$$

where the eigenvalues of L_0 are $\{\pm z_m : m = 1, \ldots, n\}$.

If the $c_k = 1$ for all k, then $\tilde{T}$ is symmetric and no blow up is possible. We will restrict to this case from now on. We appeal to the abundant literature on the Toda lattice for the results needed [8, 17, 18]. In particular, under mild assumptions, as $t_{2j} \to \pm\infty$, $T(t_{2j})$ given by (3.11) converges to a diagonal matrix. As a result we see that $a_k(t_{2j}) \to 0$.

In terms of the Pfaff lattice this is showing that $L(t_2)$ converges to a 2×2 block diagonal matrix as $t_2 \to \infty$ where the jth diagonal block has the form

$$\begin{pmatrix} 0 & 1 \\ z_j^2 & 0 \end{pmatrix}.$$

The blocks will be sorted by the size of z^2 so that the blocks with pairs of imaginary eigenvalues will appear in the lower right corner while those with pairs of real eigenvalues will appear in the upper left corner. For the t_4-flow, the eigenvalues will again be such a pair but will now be sorted by the size of z^4, mixing the blocks with real and imaginary pairs.

While the end result is that the Pfaff lattice restricted to symplectic lower Hessenberg matrices is equivalent in every sense to the indefinite Toda lattice flows, this is not a result which is immediately obvious from the Pfaff lattice itself. One sees that even in the case when the $c_k = 1$ for all k, the long time dynamics of the Pfaff lattice flows of symplectic lower Hessenberg matrices is significantly different from those of non-symplectic lower Hessenberg matrices; for example the the symplectic matrices are in fact fixed points of the flow. This difference alone merited a separate work on these cases and produced some interesting and new connections to generalizations of the Toda lattice.

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Appendix. SR-algorithm

The Toda lattice may be viewed as a continuous version of the QR-algorithm for diagonalizing a symmetric tridiagonal matrix. In the same way, the Pfaff lattice is a continuous version of the SR-algorithm on an H^S-tridiagonal matrix. Here we collect some pertinent details about the SR-algorithm. These results have been covered from a variety of points of view; a partial list of references is [4, 7, 9, 19–23].

Here we consider the SR-algorithm for a symplectic matrix which is symplectically similar to a lower Hessenberg matrix. Recall that any regular matrix can be reduced to a similar Hessenberg matrix by Householder's method, a fact used in giving an efficient version of the QR-algorithm. However, note that the Householder's method does not, in general, give a symplectic conjugation.

Let s_1 be a row vector in $\mathbb{R}^{2n}$ and h be a $2n \times 2n$ matrix; then we define the $2n \times 2n$ matrix

$$K(h, s_1) = \begin{pmatrix} s_1 \\ s_1 h \\ s_1 h^2 \\ \vdots \\ s_1 h^{2n-1} \end{pmatrix},$$

so that $K(h, s_1)$ satisfies: $K(h, s_1)h = C_h K(h, s_1)$. We will then need theorem 3.4 from [7]:

Theorem A.1. *Let $h \in Sp(n, \mathbb{R})$ and s_1 be a row vector in $\mathbb{R}^{2n}$. Then there exists a symplectic transformation S such that ShS^{-1} is a lower Hessenberg symplectic matrix iff $K(h, s_1)$ has an SR-factorization $K(h, s_1) = RS$, where R is a lower triangular matrix and s_1 is the first row of $S \in Sp(n, \mathbb{R})$. In addition, if this factorization exists, then*

$$L_0 := ShS^{-1} = R^{-1} C_h R$$

is in the lower Hessenberg form defined in (1.6).

A dense set of symplectic matrices h may be placed in this form.

Furthermore, at the expense of excluding a large set of symplectic matrices, we may refine theorem A.1:

Theorem A.2. *Let $h \in Sp(n, \mathbb{R})$, and s_1 be a row vector in $\mathbb{R}^{2n}$. Then there exists a symplectic transformation S such that ShS^{-1} is a lower Hessenberg matrix iff $K(h, s_1)$ has*

an SR-factorization $K(h, s_1) = RS$ with $R \in G_{\mathfrak{k}}$, and with s_1 equal to the first row of S. In addition, if this factorization exists, then

$$L_0 := ShS^{-1} = R^{-1}C_h R$$

is in an H^S-tridiagonal form (1.6) with $c_k = 1$ for all k.

We now show that if h has quadruples of complex eigenvalues then it will not be similar to a matrix in the H^S-tridiagonal form of (1.6) with $c_k = 1$ for all k:

Proposition A.3. *Let L be a $2n \times 2n$ matrix in the form of (1.6) with $c_k = 1$ for all k. The eigenvalues of L come in real or imaginary pairs without multiplicity.*

Proof. One checks that for L in this form, we have

$$L^2 = T \otimes I_2,$$

where T is a symmetric tridiagonal $n \times n$ matrix given by

$$T = \begin{pmatrix} d_1 & a_1 & 0 & \cdots & 0 \\ a_1 & d_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & a_{n-1} & d_n \end{pmatrix}. \tag{A.1}$$

From this one computes that

$$F(\lambda) = \det(L^2 - \lambda I_{2n}) = [\det(T - \lambda I_n)]^2 .$$

As T is symmetric, $F(\lambda)$ only has n real roots each with multiplicity 2; therefore the eigenvalues of L are square roots of real numbers, and so are only real or imaginary. □

There is an algorithm for carrying out the transformations of theorems A.1 and A.2, see [9].

This procedure is the analogous operation to the QR-algorithm step of transforming a symmetric matrix to a tridiagonal form by the Householder's method. There are large time savings in carrying out this transformation first as an SR-factorization of a matrix in the form of L_0 only requires $\mathcal{O}(n)$ operations.

The SR-algorithm is defined as the following iteration algorithm for a symplectic matrix L_0: First, factor $L_0 = R_1 S_1$ with $R_1 \in G_{\mathfrak{k}}$ and $S_1 \in Sp(n, \mathbb{R})$. Then set $L_1 = S_1 R_1$ which can be written as $L_1 = S_1 L_0 S_1^{-1} = R_1^{-1} L_0 R_1$. The iteration is then defined by $L_{k-1} \to L_k = S_k L_{k-1} S_k^{-1} = R_k^{-1} L_{k-1} R_k$ for $k = 1, 2, \ldots$. As each step is a similarity transform of the previous step by both a matrix in the symplectic group and a matrix in $G_{\mathfrak{k}}$, L_k is still a lower Hessenberg matrix which is also symplectic, and therefore is still in the H^S-tridiagonal form of L_0 above.

If the algorithm is successful the L_k approaches a family of block diagonal matrices with blocks of the form

(a) 2×2 blocks containing two real eigenvalues $(z, -z)$,
(b) 2×2 blocks containing two imaginary eigenvalues $(z, -z)$,
(c) 4×4 blocks containing a quadruple of complex eigenvalues $(z, \bar{z}, -z, -\bar{z})$.

In addition the blocks are sorted by the size of $\ln|z|$. If there are complex eigenvalues the sequence does not converge to a fixed matrix, rather it approaches the sorted diagonal shape. If the $c_j = 1$ for all j, then L_k converges to a block diagonal matrix with just 2×2 blocks. The rate of convergence is $\mathcal{O}(k^3)$ for a dense set of L_0. In practice one runs the algorithm until

$\sup_j |a_j(k)|$ at the kth step is less than some fixed ϵ tolerance. The algorithm also works on the initial matrix h (i.e. without the change to a lower Hessenberg L_0).

There is a substantial literature on improvements to this basic algorithm using implicit SR-factorization steps on certain matrix functions of L_k rather than just L_k (see e.g. [9] and references therein).

We now show that the SR-algorithm is directly equivalent to the Pfaff lattice flows with a non-traditional Hamiltonian. In light of the connection to the indefinite Toda lattice and HR-algorithms, this is not a surprising fact and the proof is appropriately close to that for the analogous fact in the Toda cases.

Proposition A.4. *Let $L_0 \in \mathfrak{sp}(n)$ in the form of (1.5). Then the SR-algorithm is equal to the integer evaluations of the Pfaff lattice flow with respect to Hamiltonian $H(L) = \mathrm{tr}(L\ln(L) - L)$ with $L(0) = L_0$.*

Proof. Recall that the SR-algorithm for the initial matrix L_0 is given by

$$L_{k-1} = R_k S_k \qquad \text{and} \qquad L_k := S_k R_k.$$

One can see that

$$L_k = S_k S_{k-1} \cdots S_2 S_1 L_0 S_1^{-1} S_2^{-1} \cdots S_{k-1}^{-1} S_k^{-1}.$$

While the Pfaff flows arise from (3.3),

$$L(t) = S(t) L_0 S(t)^{-1},$$

where

$$\exp(t\ln(L_0)) = L_0^t = R(t)S(t).$$

We want to show that $S(k) = S_k S_{k-1} \cdots S_2 S_1$. We prove this by induction. First we check that $S(1) = S_1$, from which $L_1 = L(1)$. Next we make the inductive hypothesis that $S(k-1) = S_{k-1} S_{k-2} \cdots S_2 S_1$ and then consider

$$\begin{aligned} R(k)S(k) &= \exp(k\ln(L_0)) = L_0^k = L_0^{k-1} L_0 \\ &= R(k-1)S(k-1)L_0 \\ &= R(k-1)L_{k-1}S(k-1) \\ &= R(k-1)R_k S_k S(k-1). \end{aligned}$$

By uniqueness of SR-factorizations for R in the identity component of $G_{\mathfrak{k}}$, we see that $S(k) = S_k S(k-1) = S_k S_{k-1} \cdots S_2 S_1$. □

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